Distributing Many Points on a Sphere

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The problem of distributing a large number of points uniformly *over the surface of a sphere has not only inspired mathematical researchers, it has attracted the attention of biologists, chemists, and physicists working in such fields as viral morphology, crystallography, molecular structure, and electrostatics. In two dimensions, the analogous problem is simply that of uniformly distributing points on the circumference of a disk, and equally spaced points provide an obvious answer. So we are faced with this question: What sets of points on the sphere imitate the role of the roots of unity on the unit circle?*

One way such points can be generated is via optimization with respect to a suitable criterion such as "generalized energy." Although there is a large and growing literature concerning such optimal spherical configurations of N points when N is "small," here we shall focus on this question from an asymptotic perspective ($N \to \infty$).

The discovery of stable carbon-60 molecules (Kroto, *et al.*, 1985)* with atoms arranged in a spherical (soccer ball) pattern has had a considerable influence on current scientific pursuits. The study of this C_{60} buckminsterfullerene also has an elegant mathematical component, revealed by F.R.K. Chung, B. Kostant, and S. Sternberg [5]. Now the search is on for much larger stable carbon molecules! Although such molecules are not expected to have a strictly spherical structure (due to bonding constraints), the construction of large stable configurations of spherical points is of interest here, as an initial step in hypothesizing more complicated molecular net structures.

In electrostatics, locating identical point charges on the sphere so that they are in equilibrium with respect to a Coulomb potential law is a challenging problem, sometimes referred to as the dual problem for stable molecules.

Certainly, uniformly distributing many points on the sphere has important applications to the field of computation. Indeed, quadrature formulas rely on appropriately chosen sampled data-points in order to approximate area integrals by taking averages in these points. Another example arises in the study of computational complexity, where M. Shub and S. Smale [22] encountered the problem of determining spherical points that maximize the product of their mutual distances.

These various points of view clearly lead to different extremal conditions imposed on the distribution of N points. Except for some special values of N (e.g., $N = 2, 3, 6, 12, 24$) these various conditions yield different optimal configurations. However, and this is the main theme of this article, the general pattern for optimal configurations is the same: points (for N large) appear to arrange themselves according to a hexagonal pattern that is slightly perturbed in order to fit on the sphere.

To make this more precise, we introduce some notation. We denote by S^2 the unit sphere in the Euclidean space $\mathbf{R}^3$:

$$S^2 = \{x \in \mathbf{R}^3 : |x| = 1\}.$$

Lebesgue (area) measure on S^2 is denoted by σ, so that

*Curl, Kroto, and Smalley received the 1996 Nobel Prize in Chemistry for their work on fullerenes.

$\sigma(S^2) = 4\pi$. A generic subset of S^2 with N elements will be denoted by ω_N. Associated with a configuration $\omega_N = \{x_1, x_2, \ldots, x_N\}$ is a partitioning of the sphere into Dirichlet (Voronoi) cells $D_1, \ldots, D_N$:

$$D_j := \{x \in S^2 : |x - x_j| = \min_{1 \le k \le N} |x - x_k|\}, \quad j = 1, \ldots, N.$$

These Dirichlet cells are closed subsets of S^2 satisfying

$$\bigcup_{j=1}^{N} D_j = S^2, \quad D_j \cap D_k \text{ has empty interior if } j \neq k.$$

For large numbers of points, we have observed experimentally (numerically) that all but exactly 12 of the Dirichlet cells for an optimal configuration are *hexagonal.* The exceptional cells are *pentagons.*

Why Hexagons?

In the plane, we have the regular hexagonal tiling, and the centers and vertices of such hexagons solve several important extremal problems in the plane. One example is the *best-packing problem:* Arrange nonoverlapping disks of the same fixed radius so that the number of disks per unit area is maximized. Likewise, the so-called *best-covering problem* is solved by the same points generated from the hexagonal tiling.

We cannot tile the sphere using only hexagons. This is a consequence of the formula for the Euler characteristic: $F - E + V = 2$, where F is the number of faces, E is the number of edges, and V is the number of vertices. A modification of the hexagonal pattern such as the addition of some pentagonal faces is needed for spherical tessellation. Again, from the Euler characteristic, it follows that any tiling of the sphere consisting exclusively of hexagons and pentagons must have exactly 12 pentagons (assuming exactly 3 edges emanate from each vertex). These 12 pentagons may be viewed as the deformations (perturbations) in the hexagonal pattern, enabling it to fit on the sphere.

As an example, we have for $N = 32$ the familiar soccer ball design, which has 12 pentagonal and 20 hexagonal faces. The dual structure of the soccer ball consisting of carbon atoms placed at the vertices of the faces is the previously mentioned C_{60} molecule. Another, rather ancient and more complicated example is depicted on the cover of *The Mathematical Intelligencer*, vol. 17, no. 3 (summer 1995), as the tiling of the ball beneath the lion's paw in a statue guarding the gates of a Chinese palace (finding the pentagonal faces amongst the many hexagonal ones there is an amusing diversion).

Best Packing on the Sphere

This is one of the most studied problems in the mathematical literature dealing with spherical point arrangements. It is also referred to as *Tammes's problem*, or the *hard-spheres problem.* It asks us to maximize the smallest distance among N points on the sphere. (For background, see, e.g., [6].)

Asymptotic results on

$$d_N := \max_{x_1, \ldots, x_N \in S^2} \min_{1 \le j < k \le N} |x_j - x_k|$$

were obtained first by W. Habicht and B. L. van der Waerden [13]. They proved that for some constant $C > 0$,

$$\left(\frac{8\pi}{\sqrt{3}}\right)^{1/2} N^{-1/2} - CN^{-2/3} \le d_N \le \left(\frac{8\pi}{\sqrt{3}}\right)^{1/2} N^{-1/2}.$$

The essential idea for establishing the lower bound is to project the hexagonal tiling of the plane onto the sphere, from which the dominant term

$$\delta_N := \left(\frac{8\pi}{\sqrt{3}}\right)^{1/2} N^{-1/2}$$

can be seen to arise via the following heuristic argument.

For a planar hexagonal lattice normalized so that the minimal distance between points is 1,

$$\{m + ne^{i\pi/3} : m, n \in \mathbb{Z}\},$$

the Dirichlet cell of each lattice point is a hexagon with area $\sqrt{3}/2$. Considering an optimal arrangement of N points on the sphere, we now imagine that a typical point is the center of a Dirichlet cell whose projection on a tangent plane is part of a suitably scaled hexagonal lattice. (Here, we ignore the 12 pentagonal cells.) The required scaling factor δ_N is obtained by equating total areas:

$$N \frac{\sqrt{3}}{2} \delta_N^2 = 4\pi,$$

which gives the value $\delta_N = (8\pi/\sqrt{3})^{1/2} N^{-1/2}$ for the approximate minimal distance between points.

The technique of hexagonal projections has also been recently employed by the authors to study a general class of minimal-energy problems that we now describe.

Extremal Energy

Fekete points $x_1^*, x_2^*, \ldots, x_N^*$ on the sphere are points that minimize

$$E(1, \omega_N) := \sum_{1 \le j < k \le N} |x_j - x_k|^{-1}.$$

Physically, this represents the energy of N charged particles that repel each other according to Coulomb's law. More generally, one may consider points that minimize the *s-energy*

$$E(s, \omega_N) := \sum_{1 \le j < k \le N} |x_j - x_k|^{-s}, \quad s > 0.$$

As $s \to \infty$, with N fixed, the s-energy is increasingly dominated by the term involving the smallest of the distances. In this sense, the problem leads to the best-packing problem.

Of related interest are points that maximize the product of the distances

$$\prod_{1 \le j < k \le N} |x_j - x_k|.$$

This is equivalent to minimizing the logarithmic energy

$$E(0, \omega_N) := \sum_{1 \le j < k \le N} \log \frac{1}{|x_j - x_k|},$$

and extremal points for this problem are called *logarithmic extreme points* or *elliptic Fekete points.*

Finally, we mention the criterion of maximizing the sum of powers of the distances

$$\sum_{1\le j<k\le N} |x_j - x_k|^\alpha, \quad \alpha > 0.$$

This problem is only interesting for $\alpha < 2$; for even N and $\alpha \ge 2$, an extremal configuration is obtained by placing half the points at the north pole and the other half at the south pole.

Extensive computations for optimal configurations and their corresponding extremal energies have been reported in a number of articles. Most deal with the Coulomb case ($s = 1$) [10,12], but other values have also been considered [4,19,21]. A particularly convenient listing is provided by Hardin, Sloane, and Smith, who have made their findings accessible via the internet address *netlib.att.com.* For large numbers N (say, $N \ge 100$), these computations are far from trivial, and in fact are sometimes used for testing global optimization routines [2].

The bad (yet fascinating) news arising from computations is that there are many local minima for the s-energy problem that are not global minima. In addition, the local minima have energies very close to the global minimum. These facts make it very difficult to determine the precise minimum. It is estimated that the number of distinct local minima (ignoring rotations and reflections of the sphere) grows exponentially with N (see [11]).

On the other hand, the good news is that, with only a handful of exceptions, the structures of extremal configurations computed for $32 \le N \le 200$ with $s = 0$ and $s = 1$ do indeed exhibit the hexagonal–pentagonal pattern of Dirichlet cells. See, for example, Figure 1, which shows 122 points in equilibrium along with their Dirichlet cells for the case $s = 1$. Its dual structure (the cage formed by the cells), which is displayed in Figure 2, provides a spherical candidate for the network pattern of carbon-240 molecules. All of the figures in this article were provided by Yanmu Zhou and appear along with extensive tables in his thesis [26].

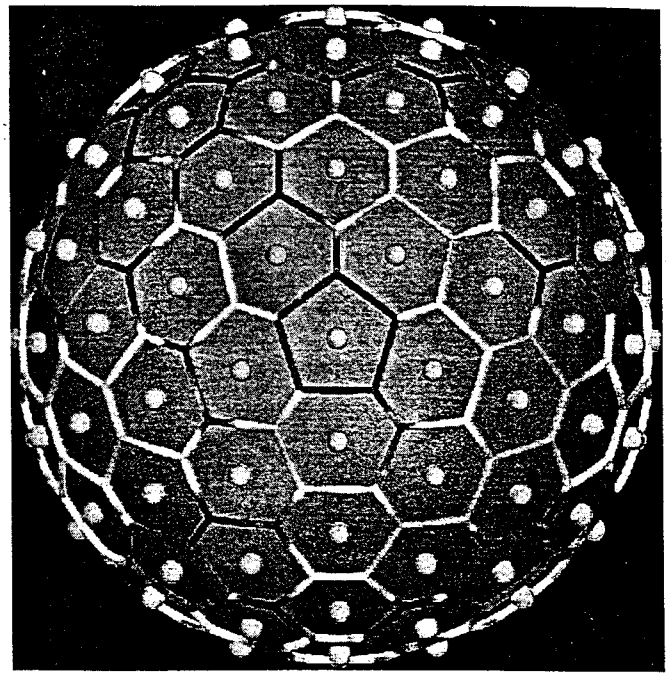

Figure 1. The 122 electrons in equilibrium and their Dirichlet cells.

Asymptotics for Extremal Energies

Although the precise determination of optimal configurations for large N seems remote at best, explicit constructions of configurations that are close to optimal may well be within reach. A first step toward this goal is to determine precise asymptotics for the extremal energies. Results in this direction are due to G. Wagner, E.A. Rakhmanov, E.B. Saff, and Y.M. Zhou (for $0 < s < 2$ and for logarithmic extremal points), and the authors (for $s > 0$). R. Alexander [1], J. Beck [3], and K. B. Stolarsky [23] have obtained results for the maximal distance sums.

Denote by $\mathcal{E}(s, N)$ the minimal energies:

$$\mathcal{E}(s, N) := \min\{E(s, \omega_N) : \omega_N \subset S^2\}, \quad s \ge 0.$$

For $0 < s < 2$, the energy integral

$$I(s) = \frac{1}{(4\pi)^2} \int\int_{S^2\times S^2} \frac{1}{|x-y|^s}\, d\sigma(x)\, d\sigma(y) \qquad (1)$$

is finite; indeed, a simple calculation shows its value to be $2^{1-s}/(2-s)$. This enables the direct use of potential-theoretic tools. Wagner [24,25] and Rakhmanov, Saff, and Zhou [20] have thereby shown that

$$\mathcal{E}(s, N) = \frac{2^{-s}}{2-s} N^2 - R_{N,s}, \quad 0 < s < 2, N \ge 2, \qquad (2)$$

where for some positive constants $C_1 = C_1(s)$ and $C_2 = C_2(s)$,

$$C_1 N^{1+s/2} \le R_{N,s} \le C_2 N^{1+s/2}. \qquad (3)$$

Thus, the dominant term in the energy is provided by $N^2/2$ times the energy $I(s)$ for the uniform distribution. The case of minimal logarithmic energy (which in a limiting sense corresponds to $s = 0$) can be likewise treated, yielding

$$\mathcal{E}(0, N) = \frac{-1}{4} \log\left(\frac{4}{e}\right) N^2 - \frac{1}{4} N \log N - R_{N,0}, \quad N \ge 2, \qquad (4)$$

with

$$C_1 N \le R_{N,0} \le C_2 N.$$

The challenging next step is to determine whether the limit

$$\lim_{N\to\infty} \frac{R_{N,s}}{N^{1+s/2}}$$

actually exists (for each fixed $s \ge 0$). For the case $s = 1$ of Fekete points, we conjecture that it does exist and equals

$$-3\left(\frac{\sqrt{3}}{8\pi}\right)^{1/2} \zeta\left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{3n+1}} - \frac{1}{\sqrt{3n+2}}\right)$$
$$\approx 0.55305\ldots,$$

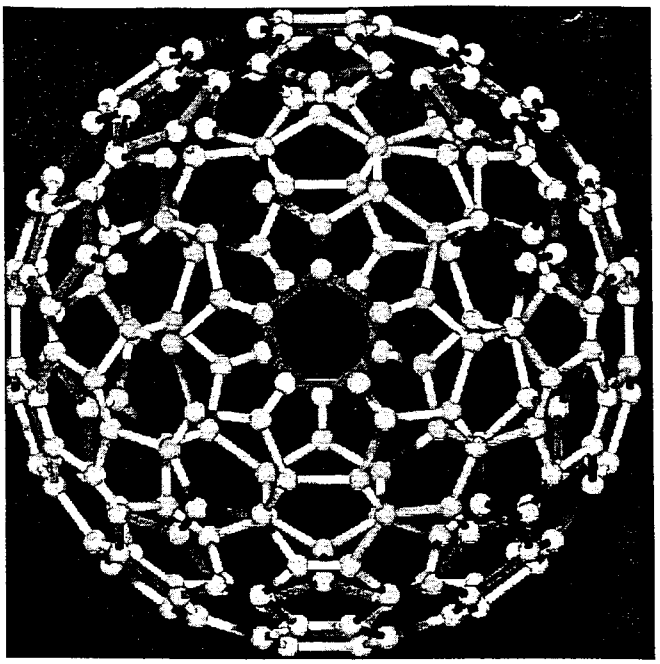

Figure 2. Dual structure for 122 electrons in equilibrium.

which involves the value of the Riemann zeta function at 1/2. This conjectured limit is again derived from considerations of the hexagonal–pentagonal tiling of the sphere (see [17]). Computational results up to 200 points give a value close to 0.5523 [12,20].

For $s \geq 2$, the energy integral (1) is infinite and the representation (2) no longer meaningful because the remainder becomes dominant. This transition in asymptotic behavior is likely due to the fact that for $s \geq 2$, the influence on the energy of "nearby points" starts to grow substantially. In fact, for $s = 2$, the authors [17] have shown (using spherical harmonics) that

$$\lim_{N\to\infty} \frac{\mathcal{E}(2, N)}{N^2 \log N} = \frac{1}{8},$$

and for $s > 2$,

$$C_1 N^{1+s/2} \leq \mathcal{E}(s, N) \leq C_2 N^{1+s/2},$$

where C_1 and C_2 are positive constants (depending on s). Moreover, a hexagonal tiling projection technique yields the upper estimate

$$\limsup_{N\to\infty} N^{-1-s/2}\, \mathcal{E}(s, N) \leq \frac{1}{2}\left(\frac{\sqrt{3}}{8\pi}\right)^{s/2} \zeta_L(s), \quad s > 2, \quad (5)$$

where $\zeta_L(s)$ is the zeta function of the quadratic number field $\mathbf{Q}(\sqrt{-3})$. Also, we have conjectured that inequality (5) holds with equality for the ordinary limit.

Equal-Area Partitioning

For various purposes it is useful to have a partitioning of the sphere into N equal-area parts with small diameters. It is not difficult to see that the optimal diameter lengths should have order $N^{-1/2}$.

In this direction, we quote the following result of Rakhmanov–Saff–Zhou [20,26]:

For every $N \geq 2$, there are subsets $D_1, \ldots, D_N$ of S^2 such that

$$\begin{aligned} &\bigcup_{j=1}^{N} D_j = S^2, \quad D_j \cap D_k \text{ has empty interior if } j \neq k, \\ &\sigma(D_j) = 4\pi/N, \quad \operatorname{diam}(D_j) \leq 7/\sqrt{N}, \quad j = 1, \ldots, N. \end{aligned} \quad (6)$$

The essential feature here is the sufficiency of the constant 7 appearing in the diameter estimate. Although it is not sharp (in fact, the value 6 suffices for $N \geq 3100$), it cannot be replaced by a constant less than 4. (The best asymptotic constant is not known.) Figure 3 illustrates this partitioning for $N = 400$.

To show the usefulness of such a partition, we prove the lower bound of Inequality (3); that is, $R_{N,s} \geq C_1 N^{1+s/2}$ for $0 < s < 2$. This argument is much simpler than the original one by Wagner and yields a numerical value for the constant C_1.

Using the sets in Eqs. (6), we put $d\sigma^* := (N/4\pi)d\sigma$ and $d\sigma_i^* := d\sigma^*|_{D_i}$. Then choosing the points $x_1, \ldots, x_N$ randomly and independently from $D_1, \ldots, D_N$, respectively, we get for the expected value of the energy

$$\begin{aligned} &\int\int\cdots\int \sum_{1\leq i<j\leq N} \frac{1}{|x_i - x_j|^s}\, d\sigma_1^*(x_1)\, d\sigma_2^*(x_2) \cdots d\sigma_N^*(x_N) \\ &= \frac{1}{2} \sum_{i\neq j} \int_{D_i}\int_{D_j} \frac{1}{|x-y|^s}\, d\sigma_i^*(x)\, d\sigma_j^*(y) \\ &= \frac{1}{2}\int\int \frac{1}{|x-y|^s}\, d\sigma^*(x)\, d\sigma^*(y) \\ &\quad - \sum_{i=1}^{N} \int_{D_i}\int_{D_i} \frac{1}{|x-y|^s}\, d\sigma_i^*(x)\, d\sigma_i^*(y) \\ &\leq \frac{I(s)}{2} N^2 - \sum_{i=1}^{N} \frac{1}{[\operatorname{diam}(D_i)]^s} \leq \frac{I(s)}{2} N^2 - \frac{1}{7^s} N^{1+s/2}. \end{aligned}$$

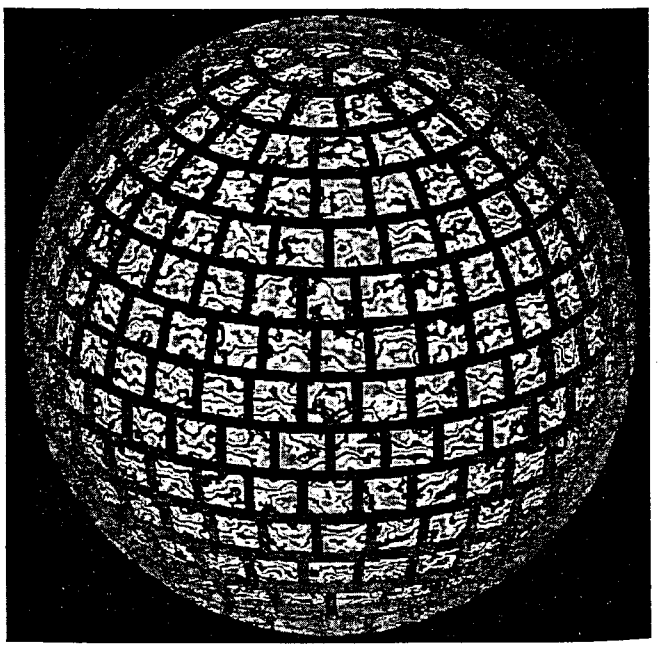

Figure 3. Partition of the sphere into 400 equal-area parts with diameters $\leq 7/\sqrt{400}$

Since the expected value is bounded by $[I(s)/2]N^2 - (1/7^s)N^{1+s/2}$, there is also a configuration that meets this bound. Consequently, Inequality (3) holds with $C_1 = 1/7^s$.

Without the equal-area requirement, partitioning the sphere into N parts of small diameters is a problem related to best packing; that is, to the optimal separation of points on the sphere. In fact, the previously mentioned result of Habicht and van der Waerden resulting from hexagonal tiling projections leads to an asymptotic order of

$$\frac{4}{\sqrt{N}}\sqrt{\frac{2\pi}{\sqrt{27}}} \approx \frac{4.398}{\sqrt{N}}$$

for an attainable size of the diameters.

Certainly, we expect that optimal s-energy configurations possess a "well-separation" property; that is, the distance between any pair of points of the configuration should be bounded below by $C/\sqrt{N}$. At this writing, such results have been established for $s = 0$ [21], $s = 1$ [8], and $s > 2$ [17].

Separation properties also play an important role in the analysis of quadrature formulas, as we now explain.

Spherical Designs

Of practical importance is the use of uniformly distributed points on the sphere for numerical integration purposes. For example, in the analysis of satellite data from the earth's surface, one frequently wants to approximate integrals over a sphere by arithmetic averages at some well-chosen points. The mathematical problem is to choose configurations $\omega_N = \{x_{1,N}, \ldots, x_{N,N}\}$ (typically N is very big) such that the difference

$$\left| \frac{1}{4\pi}\int_{S^2} f(x)\, d\sigma(x) - \frac{1}{N}\sum_{j=1}^{N} f(x_{j,N}) \right|$$

is small (maybe even zero) for a large class of functions. Various classes of functions have been considered in the literature. As an example we mention the class of indicator functions of spherical caps which leads to the notion of spherical cap discrepancy.

All considered classes give rise to a sequence of configurations ω_N that are *asymptotically uniformly distributed* in the sense that for every spherical cap A with area $\sigma(A)$

$$\lim_{N\to\infty} \frac{\#\{1 \le j \le N : x_{j,N} \in A\}}{N} = \frac{1}{4\pi}\sigma(A),$$

which means that for large N, each spherical cap gets its fair share of points. This is equivalent to the property that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} f(x_{j,N}) = \frac{1}{4\pi}\int_{S^2} f(x)\, d\sigma(x)$$

for every continuous function f on S^2.

When dealing with smooth functions, it is natural to consider the class of polynomials up to a certain degree. This leads to the notion of spherical designs, which was introduced in 1977 by P. Delsarte, J. M. Goethals, and J. J. Seidel [9], and is so named because of certain analogies with combinatorial designs. By definition, a configuration $\omega_N = \{x_1, \ldots, x_N\}$ is a *spherical t-design* if

$$\frac{1}{4\pi}\int_{S^2} f(x)\, d\sigma(x) = \frac{1}{N}\sum_{j=1}^{N} f(x_j) \qquad (7)$$

for all polynomials f in three variables of degree $\le t$. Spherical t-designs with small N and large t provide excellent integration formulas for smooth functions. However, there seem to be no explicit constructions available that work for large t. Extensive computer searches by Hardin and Sloane [14] have led to the discovery of spherical t-designs for t up to 13 with conjectured minimal number of points. For example, they found a spherical 13-design with 94 points.

We use $N(t)$ to denote the minimum number of points for which a spherical t-design exists. From our asymptotic point of view, we want to study the behavior of $N(t)$ as $t \to \infty$. It is not so hard to show that $N(t) \ge Ct^2$. This may be viewed as a natural bound, since S^2 is a two-dimensional algebraic variety. The space of polynomials of degree $\le t$ restricted to the sphere has dimension $(t+1)^2$. So the requirements for a spherical design constitute $(t+1)^2$ equations, and from this, one can show that one needs at least Ct^2 points. A rather difficult problem seems to be the following.

Prove (or disprove): $N(t) = O(t^2)$ for $t \to \infty$.

Hardin and Sloane [14] even conjecture that $N(t) \le \frac{1}{2}t^2(1 + o(1))$. The best result so far is due to J. Korevaar and J. L. H. Meyers [16]. They proved $N(t) = O(t^3)$. An interesting relation with extremal energies was pointed out by Korevaar [15]. Introducing the modified Coulomb energy

$$E(1, r; \omega_N) := \sum_{1\le j<k\le N} \frac{1}{|x_j - rx_k|},$$

where r is a parameter, $0 < r < 1$, he considers the minimizing configurations $\omega_N^{(r)}$. It follows from Korevaar's results that if the minimizing configurations are well separated [i.e., if there is a constant $C > 0$ such that

$$|x_j^{(r)} - x_k^{(r)}| \ge \frac{C}{\sqrt{N}}, \qquad 1 \le j < k \le N,$$

for all N and all $r \in (0, 1)$], then $N(t) = O(t^2)$ as $t \to \infty$. This approach clearly merits further study.

Spiral Points

A variety of algorithms has been proposed for the explicit construction of uniformly distributed points on the sphere. Some begin with uniformly distributed points over a rectangular area which are then transformed via cylindrical projection onto the sphere. Others employ sequences of rotations of the sphere which, under stereographic projection, correspond to Möbius transformations in the plane (cf. [18]).

A popular and simple technique is that of icosahedral dissection, which proceeds as follows: For each triangular face of the icosahedron, the midpoints of the sides are joined to form four new triangles. The centers of the new

triangles are then radially projected onto the sphere, yielding a total of 80 points. Continuing the dissection process produces $N = 20 \cdot 4^n$ points, $n = 0, 1, \ldots$. One apparent drawback of this method is the restricted values for N. A worse fault is their bad performance with respect to the energy criteria discussed earlier. What's more, they are *not* asymptotically uniformly distributed. To see this, it is enough to observe that after the first step, the points are centered at 80 spherical triangles which do not all have the same area (the projection process increases the areas of the "middle triangles" more than the rest). As the further steps in the process yield the same number of points in each of the 80 triangles, asymptotic uniform distribution cannot hold. The lackluster behavior of these points was also detected by J. Cui and W. Freeden [7], who have developed a generalized discrepancy test for comparing the proportion of points in spherical regions with the normalized areas of the regions.

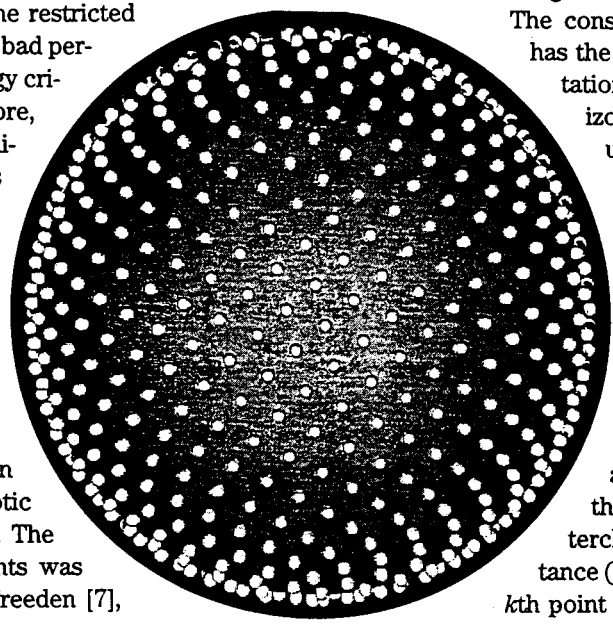

Figure 4. Generalized spiral points for N = 700.

Motivated by hexagonal tilings and numerical experimentation, Rakhmanov, Saff, and Zhou [20] have recently introduced the following simple construction, which appears for large N to have a considerable advantage over the above-mentioned algorithms. Using spherical coordinates (θ, ϕ), $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, we set

$$\theta_k = \arccos(h_k), \qquad h_k = -1 + \frac{2(k-1)}{(N-1)}, \quad 1 \le k \le N,$$

$$\phi_k = \left(\phi_{k-1} + \frac{3.6}{\sqrt{N}} \frac{1}{\sqrt{1-h_k^2}}\right) \pmod{2\pi}, \tag{8}$$

$$2 \le k \le N-1, \ \phi_1 = \phi_N = 0.$$

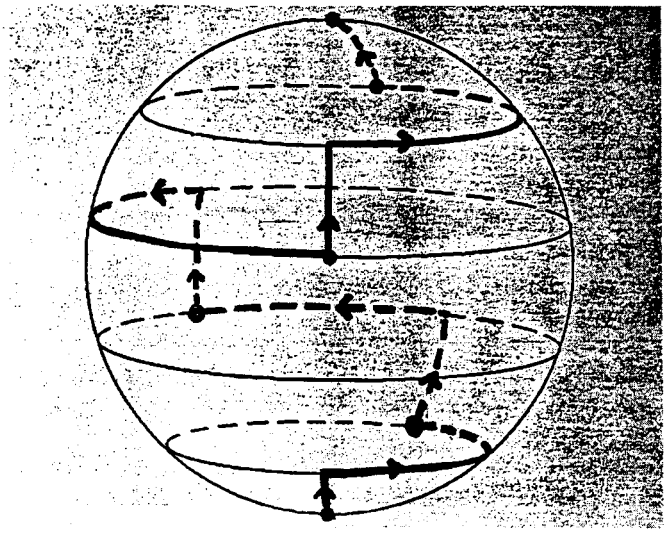

Figure 5. Construction of generalized spiral points for N = 6.

Then the point set $\hat{\omega}_N = \{(\theta_k, \phi_k)\}_{k=1}^N$ is called a *generalized spiral set* on S^2. For $N = 700$, these points are illustrated in Figure 4.

The construction of these spiral points has the following geometrical interpretation. One cuts the globe with N horizontal planes spaced $2/(N-1)$ units apart, forming N circles of latitude on the sphere [the first and last of these are the degenerate circles (points) consisting of the south and north poles]. Each latitude contains precisely one spiral point. To obtain the kth spiral point, one proceeds upward from the $(k-1)$st point $(\theta_{k-1}, \phi_{k-1})$ along a great circle (meridian) to the next latitude and travels counterclockwise along it for a fixed distance (independent of k) to arrive at the kth point (θ_k, ϕ_k), as illustrated in Figure 5. This fixed distance is the value that must be assigned to $(\phi_k - \phi_{k-1})\sqrt{1-h_k^2}$, and the result of Habicht and van der Waerden for best packing suggests that it should be close to

$$\delta_N = \left(\frac{8\pi}{\sqrt{3}}\right)^{1/2} \frac{1}{\sqrt{N}} \approx 3.809 \frac{1}{\sqrt{N}};$$

that is,

$$(\phi_k - \phi_{k-1})\sqrt{1-h_k^2} \approx \frac{3.809}{\sqrt{N}}.$$

The choice of the smaller constant 3.6 in Eqs. (8) is based on numerical experimentation and can be viewed as an accommodation to the fact that distances shrink when a hexagonal lattice is orthogonally projected from a tangent plane to the sphere. [More generally, in Eqs. (8) one can introduce a parameter C in place of 3.6 and adjust its value appropriately for the application at hand.]

At least for N up to 12,000, the generalized spiral points $\hat{\omega}_N$ have energies in reasonable agreement with representations (2) and (4). For example, on comparing computed values for the energy of spiral points with the theoretical estimates for the minimal energy when $s = 0$, it can be proved that

$$E(0, \hat{\omega}_N) - \mathscr{E}(0, N) \le 114 \log N, \quad 2 \le N \le 12{,}000,$$

and it is likely that the $\hat{\omega}_N$ actually perform much better than this estimate indicates. An asymptotic analysis of these generalized spiral points awaits further investigation.

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